

Strings in Homogeneous Background Spacetimes

Mariusz P. Dąbrowski ¹

Institute of Physics, University of Szczecin, Wielkopolska 15, 70-451 Szczecin, Poland

Arne L. Larsen ²

Institute of Physics, University of Odense, Campusvej 55, 5230 Odense M, Denmark

February 1, 2008

¹E-mail: mpdabfz@uoo.univ.szczecin.pl

²E-mail: all@fysik.ou.dk

Abstract

The string equations of motion for some homogeneous (Kantowski-Sachs, Bianchi I and Bianchi IX) background spacetimes are given, and solved explicitly in some simple cases. This is motivated by the recent developments in string cosmology, where it has been shown that, under certain circumstances, such spacetimes appear as string-vacua.

Both tensile and null strings are considered. Generally, it is much simpler to solve for the null strings since then we deal with the null geodesic equations of General Relativity plus some additional constraints.

We consider in detail an ansatz corresponding to circular strings, and we discuss the possibility of using an elliptic-shape string ansatz in the case of homogeneous (but anisotropic) backgrounds.

1 Introduction

There have been quite a lot of interest during the last few years in the evolution of strings in fixed curved backgrounds; for reviews, see for instance [1]. It is well-known that the string equations of motion in curved spacetimes form a complicated system of second-order non-linear coupled partial differential equations which, in general, is non-integrable. However, there are some special string configurations for which the equations are exactly solvable (see for instance [1] and references therein). A lot of such explicit solutions have already been found, but they were restricted to just the most symmetric spacetimes such like Minkowski, de Sitter, anti-de Sitter, Schwarzschild and Robertson-Walker ones.

We shall consider both tensile and null (tensionless) strings. Generally, it is much simpler to solve for the null strings [2], since then we deal with the null geodesic equations of General Relativity plus some additional constraints. It might seem then, that the dynamics of null strings is quite trivial. However, this is not true. Although each individual point along the null string follows a null geodesic, the null string as a whole may experience highly non-trivial dynamics [3]. The situation is qualitatively similar to that of congruences in General Relativity, that is, "bundles of rays": each ray in the bundle is just following a geodesic, but the propagation of the bundle as a whole can be highly non-trivial due to tidal forces, as described by the Raychaudhuri equation.

In this paper we extend the discussion of the tensile and null string evolution to the homogeneous, but anisotropic, spacetimes of Kantowski-Sachs, Bianchi type I and IX. In fact, Kantowski-Sachs solutions with negative and zero curvature are just axisymmetric Bianchi type I and III Universes. This means that only positive curvature Kantowski-Sachs models are different from Bianchi type Universes.

Our main motivation is the recent development in string-cosmology. It has been shown that Kantowski-Sachs and Bianchi type spacetimes, under certain circumstances, appear as string-vacua, i.e. as solutions to the β -function equations [4] to some (low) order in α' . Thus, in this paper, we consider the dynamics of a string moving in a condensate of the massless string modes.

The paper is organized as follows. In Sec. II, we give the tensile and null string equations of motion and constraints in generic curved spacetimes. In

Sec. III, we specialise to the case of positive curvature Kantowski-Sachs (K-S) backgrounds, in particular, we consider the K-S Universe with cosmological term and the K-S Universe with stiff-fluid matter. In both cases we solve the equations of motion for tensile and null strings after making appropriate ansätze, and we give the physical interpretation of the solutions. In Sec. IV we turn to the Bianchi Universes. We first discuss some of the problems concerning finding explicit tensile string solutions in spacetimes of Kasner type, and secondly, we give some explicit null string solutions. Finally in Sec. V, we consider circular strings in axisymmetric Bianchi IX Universes, and in Sec. VI we give our conclusions.

2 String Equations of Motion and Constraints

Let us consider the tensile and the null string equations of motion in a compact formula

$$\ddot{X}^\mu + \Gamma_{\nu\rho}^\mu \dot{X}^\nu \dot{X}^\rho = \lambda \left(X''^\mu + \Gamma_{\nu\rho}^\mu X'^\nu X'^\rho \right), \quad (2.1)$$

where dot means derivative with respect to the string coordinate τ and prime means derivative with respect to the string coordinate σ . The constraints read as

$$g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu = -\lambda g_{\mu\nu} X'^\mu X'^\nu, \quad (2.2)$$

$$g_{\mu\nu} \dot{X}^\mu X'^\nu = 0. \quad (2.3)$$

For $\lambda = 1$ we have the tensile strings while $\lambda = 0$ applies for the null strings. Notice that in Ref. [5], expansion schemes were considered, essentially using λ ($\lambda \sim 1/\alpha'$, where α' is the inverse string tension) as a continuous expansion parameter; in this paper we simply use λ as a discrete parameter discriminating between tensile and null strings.

From the above we can see that for the null strings, we have the null geodesic equation of General Relativity supplemented by the constraint (II.3). For both null and tensile strings the invariant string size is defined by (closed string)

$$S(\tau) = \int_0^{2\pi} S(\tau, \sigma) d\sigma, \quad (2.4)$$

where

$$S(\tau, \sigma) = \sqrt{-g_{\mu\nu} X'^\mu X'^\nu}. \quad (2.5)$$

3 Strings in Kantowski-Sachs Background

In this section, we will consider the string equations of motion in homogeneous Kantowski-Sachs spacetimes. The validity of the Kantowski-Sachs spacetime as being consistent string vacuum (solution to the β -function equations to the lowest order in α') is discussed elsewhere [6]. Here we just concentrate on the motion of a test string in Kantowski-Sachs backgrounds.

The Kantowski-Sachs spacetime is given by the metric [7]

$$ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)d\Omega_k^2, \quad (3.1)$$

where the "angular" metric is

$$\begin{aligned} d\Omega_k^2 &= d\theta^2 + S^2(\theta)d\varphi^2, \\ S(\theta) &= \begin{cases} \sin\theta & \text{for } k = +1, \\ \theta & \text{for } k = 0, \\ \sinh\theta & \text{for } k = -1, \end{cases} \end{aligned}$$

and A, B are the expansion scale factors. Here, $r \in]-\infty, \infty[$, while the range of t depends on the particular cosmology. For $k = +1$ the coordinates θ and φ describe, as usual, the angles on the 2-sphere. Only $k = +1$ models fall outside the Bianchi classification, but usually one refers to all three curvature models as Kantowski-Sachs Universes. In this paper, we mainly consider $k = +1$ models.

As a first example of a string configuration, we apply the following string ansatz

$$X^0 = t(\tau), \quad X^1 = r(\tau), \quad X^2 = \theta(\tau), \quad X^3 = \varphi = \sigma, \quad (3.2)$$

which describes a circular string winding around the 2-sphere. The functions $(t(\tau), r(\tau), \theta(\tau))$, which describe the dynamics of the string, are to be determined from the equations of motion.

For the metric (III.1), we start with the string equations of motion (II.1) and constraints (II.2)-(II.3), which now reduce to

$$\ddot{t} + AA_{,t}\dot{r}^2 + BB_{,t}\dot{\theta}^2 - \lambda BB_{,t}\sin^2\theta = 0, \quad (3.3)$$

$$\ddot{r} + 2\frac{A_{,t}}{A}\dot{t}\dot{r} = 0, \quad (3.4)$$

$$\ddot{\theta} + 2\frac{B_{,t}}{B}\dot{t}\dot{\theta} + \lambda \sin\theta \cos\theta = 0, \quad (3.5)$$

$$\dot{t}^2 - A^2\dot{r}^2 - B^2\dot{\theta}^2 - \lambda B^2\sin^2\theta = 0. \quad (3.6)$$

The Eq. (III.4) easily integrates to give

$$\dot{r} = \frac{dr}{d\tau} = \frac{k}{A^2}, \quad (3.7)$$

with $k = \text{const.}$. The other equations, in general, cannot be integrated, thus we must either consider special Kantowski-Sachs spacetimes or make further restrictions for the ansatz. Notice, however, that for both cases $\lambda = 0, 1$, the invariant string size is given by

$$S(\tau) = 2\pi|B(\tau)\sin\theta(\tau)|. \quad (3.8)$$

3.1 Tensile Strings

For the tensile strings, $\lambda = 1$, and Eq.(III.5) is fulfilled automatically under the assumption that $\theta = \text{const.} = \pi/2$ in (III.2). In the rest of Subsection III.A, we restrict ourselves to this case. Then after inserting (III.7) into (III.6), we obtain

$$\dot{t}^2 = \frac{k^2}{A^2} + B^2, \quad (3.9)$$

or, explicitly in terms of the string time coordinate,

$$\tau(t) = \int^t \frac{|A| dt}{\sqrt{k^2 + A^2 B^2}}, \quad (3.10)$$

while from (III.7) we get

$$r(t) = k \int^t \frac{dt}{|A|\sqrt{k^2 + A^2 B^2}}. \quad (3.11)$$

Notice also that (III.3) is automatically fulfilled now.

3.1.1 Λ -term solutions

First we refer to one of the simplest solutions given for the scale factors, which is the $k = +1$ Kantowski-Sachs Universe with only the cosmological term [8]. These are:

$$A(t) = H_0^{-1} \sinh H_0 t, \quad (3.12)$$

$$B(t) = H_0^{-1} \cosh H_0 t, \quad (3.13)$$

with $H_0 = \text{const.}$, and we consider only the expanding phase ($t \geq 0$) of the Universe. After inserting (III.12)-(III.13) into (III.10) we have (choosing boundary conditions such that $t(0) = 0$)

$$\tau(t) = H_0 \int_0^t \frac{\sinh(H_0 t)}{\sqrt{k^2 H_0^4 + \sinh^2(H_0 t) \cosh^2(H_0 t)}} dt, \quad (3.14)$$

and from (III.11) we obtain

$$r(t) = k H_0^3 \int_{t_0}^t \frac{dt}{\sinh(H_0 t) \sqrt{k^2 H_0^4 + \sinh^2(H_0 t) \cosh^2(H_0 t)}}. \quad (3.15)$$

By inverting (III.14), giving $t(\tau)$, Eq.(III.15) then gives explicitly $r(\tau)$. The Eqs.(III.14)-(III.15) can be transformed to the form of the standard elliptic integrals. For instance, by making the substitution $\cosh(H_0 t) = \sqrt{z}$, Eq.(III.14) becomes

$$\tau(z) = \frac{1}{2} \int_1^z \frac{dz}{\sqrt{z(z-z_1)(z-z_2)}}, \quad (3.16)$$

with

$$z_{1,2} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4k^2 H_0^4} \right), \quad (3.17)$$

and can thus be evaluated explicitly eventually yielding $t(\tau)$. However, the detailed form will not be important here. The invariant string size in this case is simply

$$S(\tau) = 2\pi H_0^{-1} \cosh(H_0 t(\tau)), \quad (3.18)$$

i.e., the string size follows the expansion of the Universe. This is easily understood, since the string is simply winding around the equatorial plane of the 2-sphere.

3.1.2 Time-symmetric stiff-fluid solutions

The solutions for the scale factors for the time-symmetric $k = +1$ (recollapsing) stiff-fluid Kantowski-Sachs model is given by [9]

$$A(t) = b, \quad (3.19)$$

$$B(t) = \frac{\sqrt{M}}{b} \sqrt{1 - \frac{b^2}{M}(t-t_0)^2}, \quad (3.20)$$

with (b, M, t_0) constants. The constant M appears in the density of stiff-fluid matter conservation law $\rho A^2 B^4 = M$. The Kantowski-Sachs model described by scale factors (III.19)-(III.20) begins and ends at "barrel" singularities ($A = \text{const.}$, $B = 0$) [10]. For simplicity we will take $t_0 = \sqrt{M}/b$ from now on, so that the range of t is $t \in [0, 2\sqrt{M}/b]$. For the exact solution (III.19)-(III.20), Eqs. (III.7) and (III.10) integrate to give

$$t(\tau) = \frac{\sqrt{M}}{b} + \frac{\sqrt{k^2 + M}}{b} \sin \left(\tau - \arcsin \sqrt{\frac{M}{k^2 + M}} \right), \quad (3.21)$$

$$r(\tau) = \frac{k}{b^2} \tau + r_0, \quad (3.22)$$

where we took boundary conditions such that $t(0) = 0$. Having this, one can express B in terms of τ , i.e.,

$$B(\tau) = \frac{\sqrt{M}}{b} \sqrt{1 - \frac{k^2 + M}{M} \sin^2 \left(\tau - \arcsin \sqrt{\frac{M}{k^2 + M}} \right)}, \quad (3.23)$$

which in the limit $k = 0$ ($r = \text{const.}$) gives

$$B(\tau) = \frac{\sqrt{M}}{b} \sin \tau. \quad (3.24)$$

According to (III.8), the invariant string size is simply $S(\tau) = 2\pi B(\tau)$, which means that the string trivially starts with zero size, then expands to a maximum size and finally contracts to zero size again together with the Universe, i.e. at $\tau = 2\arcsin \sqrt{M/(k^2 + M)}$. This again comes from the fact that the string simply winds around the equatorial plane of the 2-sphere.

In the next subsection, we shall consider more complicated (null) string solutions with non-trivial dynamics.

3.2 Null strings

The above solutions for tensile strings were all obtained for $\theta = \pi/2$. However, for the null strings, $\lambda = 0$, we can easily integrate (III.5), still keeping the general form of $\theta = \theta(\tau)$ in the ansatz (III.2), to obtain

$$\dot{\theta} = \frac{d\theta}{d\tau} = \frac{l}{B^2}, \quad (3.25)$$

with $l = \text{const}$. The Eq.(III.6) now becomes

$$\dot{t}^2 = \frac{k^2}{A^2} + \frac{l^2}{B^2}, \quad (3.26)$$

or, explicitly

$$\tau(t) = \int^t \frac{|AB| dt}{\sqrt{l^2 A^2 + k^2 B^2}}, \quad (3.27)$$

while (III.7) becomes

$$r(t) = k \int^t \frac{|B| dt}{|A| \sqrt{l^2 A^2 + k^2 B^2}}. \quad (3.28)$$

3.2.1 Λ -term solutions

Inserting (III.12)-(III.13) into (III.27), we have an exact relation between the spacetime and the string time coordinates

$$H_0 t(\tau) = \text{arccosh} \sqrt{\frac{l^2 + [H_0^2(k^2 + l^2)\tau + |k|]^2}{k^2 + l^2}}, \quad (3.29)$$

where we choose again boundary conditions such that $t(0) = 0$. Having given (III.29), one can write down (III.12)-(III.13) in terms of the τ -coordinate as

$$A(\tau) = \frac{1}{H_0} \sqrt{\frac{-k^2 + [H_0^2(k^2 + l^2)\tau + |k|]^2}{k^2 + l^2}}, \quad (3.30)$$

$$B(\tau) = \frac{1}{H_0} \sqrt{\frac{l^2 + [H_0^2(k^2 + l^2)\tau + |k|]^2}{k^2 + l^2}}. \quad (3.31)$$

This allows us to integrate (III.7) and (III.25) to give

$$r(\tau) = -\frac{k}{|k|} \text{arccth} \left(1 + \frac{H_0^2}{|k|} (k^2 + l^2) \tau \right) + r_0, \quad (3.32)$$

$$\theta(\tau) = -\frac{l}{|l|} \text{arctg} \left(\frac{|k|}{|l|} + \frac{H_0^2}{|l|} (k^2 + l^2) \tau \right) + \theta_0, \quad (3.33)$$

with $r_0, \theta_0 = \text{const}$. The invariant string size (III.8) is given by

$$\begin{aligned} S(\tau) &= 2\pi |B(\tau) \sin \theta(\tau)| \\ &= \frac{2\pi}{H_0 \sqrt{k^2 + l^2}} \left| (|k| + H_0^2(k^2 + l^2)\tau) \sin \theta_0 - l \cos \theta_0 \right|. \end{aligned} \quad (3.34)$$

It is useful to consider some special cases. For $l = 0$ (i.e., $\theta = \theta_0$), we have

$$A(\tau) = \sqrt{H_0^2 k^2 \tau^2 + 2|k|\tau}, \quad (3.35)$$

$$B(\tau) = H_0|k|\tau + H_0^{-1}, \quad (3.36)$$

$$t(\tau) = \frac{1}{H_0} \operatorname{arccosh}(H_0^2|k|\tau + 1), \quad (3.37)$$

$$r(\tau) = -\frac{k}{|k|} \operatorname{arccth}(H_0^2|k|\tau + 1) + r_0, \quad (3.38)$$

$$S(\tau) = \frac{2\pi}{H_0} |\sin \theta_0|(H_0^2|k|\tau + 1). \quad (3.39)$$

That is, the string winds around the 2-sphere at the angle $\theta = \theta_0$ and expands to infinite size together with the scale factor.

Another special case is given in the limit $k = 0$ (i.e., $r = r_0$), where we have

$$A(\tau) = H_0|l|\tau, \quad (3.40)$$

$$B(\tau) = \frac{1}{H_0} \sqrt{1 + H_0^4 l^2 \tau^2}, \quad (3.41)$$

$$t(\tau) = \frac{1}{H_0} \operatorname{arccosh} \sqrt{1 + H_0^4 l^2 \tau^2}, \quad (3.42)$$

$$\theta(\tau) = -\frac{l}{|l|} \operatorname{arctg}(H_0^2|l|\tau) + \theta_0, \quad (3.43)$$

$$S(\tau) = \frac{2\pi}{H_0} |H_0^2 l \tau \sin \theta_0 - \cos \theta_0|. \quad (3.44)$$

Notice that when τ goes from 0 to ∞ , the angle θ changes by $\pi/2$. Therefore we can distinguish a number of different scenarios: 1) $\theta_0 = 0$. In this case, the string starts at the equatorial plane and then moves towards one of the poles in such a way that its size $S(\tau)$ is constant, i.e., the contraction of the string is exactly balanced by the expansion of the 2-sphere. 2) $\theta_0 \neq 0$. The string starts somewhere on one hemisphere, then crosses the equator and approaches a fixed position on the other hemisphere. During its evolution, the string grows indefinitely. 3) $\theta_0 \neq 0$. The string starts somewhere on one hemisphere, then moves towards the nearest pole where it collapses. It then reappears and approaches a fixed position on the same hemisphere. After the collapse, and during its later evolution, its size will grow indefinitely.

Returning to the general expression (III.34), it is easy to see that the dynamics in the general case is qualitatively similar to the $k = 0$ case, although quantitatively different. For instance, if $\theta_0 = 0$, the string starts at a fixed angle $\theta(\tau = 0) = -\text{sign}(l)\arctg(|k|/|l|)$, and can then approach the nearest pole in such a way that its size is constant. Similarly, we can also find solutions where the string expands indefinitely, possibly after collapsing once during its early evolution.

3.2.2 Time-symmetric stiff-fluid solutions

Using (III.19)-(III.20) for the null string case, we obtain from (III.27)

$$\tau(t) = \int_0^t \sqrt{\frac{M - b^2(t - \sqrt{M}/b)^2}{l^2b^2 + k^2M(1 - b^2(t - \sqrt{M}/b)^2)/b^2}} dt, \quad (3.45)$$

where we took again boundary conditions such that $t(0) = 0$. From (III.7), we get

$$r(\tau) = \frac{k}{b^2}\tau + r_0. \quad (3.46)$$

Thus r is expressed directly as a function of τ , while (III.45) must be inverted to give $t(\tau)$. Notice that (III.45) is an elliptic integral, which can be given in terms of elliptic functions [11]

$$\begin{aligned} \tau(t) &= \frac{b}{|k|} \sqrt{\frac{M}{b^2} + \frac{l^2b^2}{k^2}} E\left(\xi, \sqrt{\frac{Mk^2}{Mk^2 + l^2b^4}}\right) - \frac{l^2b^4}{k^2\sqrt{Mk^2 + l^2b^4}} F\left(\xi, \sqrt{\frac{Mk^2}{Mk^2 + l^2b^4}}\right) \\ &\quad - \frac{b}{|k|} \left(\frac{\sqrt{M}}{b} - t \right) \sqrt{\frac{M/b^2 - (t - \sqrt{M}/b)^2}{M/b^2 + l^2b^2/k^2 - (t - \sqrt{M}/b)^2}}, \end{aligned} \quad (3.47)$$

where F, E are the elliptic integrals of first and second kind, respectively, and

$$\xi = \arcsin \left[\sqrt{\frac{Mk^2 + l^2b^4}{Mk^2}} \sqrt{\frac{M/b^2 - (t - \sqrt{M}/b)^2}{M/b^2 + l^2b^2/k^2 - (t - \sqrt{M}/b)^2}} \right]. \quad (3.48)$$

Here we analyse some special cases in which (III.47) becomes elementary. First, if $l = 0$, it gives the simple solution

$$t = \frac{|k|}{b}\tau, \quad (3.49)$$

$$r = \frac{k}{b^2} \tau + r_0, \quad (3.50)$$

$$\theta = \theta_0. \quad (3.51)$$

Then the string size is

$$S(\tau) = \frac{2\pi}{b} |\sin \theta_0| \sqrt{2|k| \sqrt{M} \tau - k^2 \tau^2}, \quad (3.52)$$

that is, the string starts with zero size, then expands and eventually recollapses together with the Universe.

Another special case is when $k = 0$, and reads as

$$\begin{aligned} \tau(t) &= \frac{M}{2|l|b^2} \left[\left(\frac{bt}{\sqrt{M}} - 1 \right) \sqrt{1 - \left(\frac{bt}{\sqrt{M}} - 1 \right)^2} + \arcsin \left(\frac{bt}{\sqrt{M}} - 1 \right) + \frac{\pi}{2} \right], \\ r &= r_0, \end{aligned} \quad (3.54)$$

$$\theta(t) = \frac{l}{|l|b} \arcsin \left(\frac{bt}{\sqrt{M}} - 1 \right) + \theta_0. \quad (3.55)$$

In this case the string size is given by

$$S(t) = \frac{2\pi\sqrt{M}}{b} \sqrt{1 - \left(\frac{bt}{\sqrt{M}} - 1 \right)^2} \left| \sqrt{1 - \left(\frac{bt}{\sqrt{M}} - 1 \right)^2} \sin \theta_0 \pm \left(\frac{bt}{\sqrt{M}} - 1 \right) \cos \theta_0 \right|. \quad (3.56)$$

If $\theta_0 = 0$, the string is at the equator and it simply follows the evolution of the Universe, i.e., it starts with zero size, then expands and eventually recollapses together with the Universe. On the other hand, if $\theta_0 \neq 0$, the string has the possibility to pass one of the poles of the 2-sphere, i.e. it starts with zero size, then expands but recollapses, then expands again and eventually recollapses together with the Universe.

It is straightforward to check that the qualitative behaviour of the string solutions in the general case (described by III.47) essentially follows the $k = 0$ case, thus we shall not go into the quantitative details here.

4 Strings in Bianchi I background

The validity of the low-energy-effective-action equations for strings in Bianchi type homogeneous spacetimes has been studied in [12], and in this section

we consider the evolution of strings based on the equations (II.1)-(II.3) in Bianchi I background spacetimes with the metric [13]

$$ds^2 = dt^2 - X^2(t)dx^2 - Y^2(t)dy^2 - Z^2(t)dz^2, \quad (4.1)$$

where X, Y, Z are the scale factors. The equations of motion and constraints are given by (II.1)-(II.3). Comparing with equations (III.1)-(III.2) in the case $\theta = \pi/2$, a natural first attempt of an ansatz is now

$$X^0 = t(\tau), \quad X^1 = x = f(\tau) \cos \sigma, \quad X^2 = y = g(\tau) \sin \sigma, \quad X^3 = z = \text{const.}, \quad (4.2)$$

and the invariant string size is

$$S(\tau) = \int_0^{2\pi} \sqrt{f^2(\tau)X^2(t(\tau)) \sin^2 \sigma + g^2(\tau)Y^2(t(\tau)) \cos^2 \sigma} d\sigma. \quad (4.3)$$

The ansatz (IV.2) describes a closed string of "elliptic-shape", in the sense that

$$\frac{x^2}{f^2} + \frac{y^2}{g^2} = 1, \quad (4.4)$$

i.e., it generalizes the circular string ansatz considered before. This seems to be the most natural ansatz in the spacetimes with the line element (IV.I) because of the shear. The equations and constraints read as

$$\ddot{t} + XX_{,t} (\dot{f}^2 \cos^2 \sigma - \lambda f^2 \sin^2 \sigma) + YY_{,t} (\dot{g}^2 \sin^2 \sigma - \lambda g^2 \cos^2 \sigma) = 0 \quad (4.5)$$

$$\ddot{f} + \lambda f + 2\frac{X_{,t}}{X}\dot{t}\dot{f} = 0 \quad (4.6)$$

$$\ddot{g} + \lambda g + 2\frac{Y_{,t}}{Y}\dot{t}\dot{g} = 0 \quad (4.7)$$

as well as

$$X^2 \dot{f} \dot{f} - Y^2 \dot{g} \dot{g} = 0, \quad (4.8)$$

$$\dot{t}^2 - X^2 (\dot{f}^2 \cos^2 \sigma + \lambda f^2 \sin^2 \sigma) - Y^2 (\dot{g}^2 \sin^2 \sigma + \lambda g^2 \cos^2 \sigma) = 0. \quad (4.9)$$

Notice, however, that the equations (IV.5)-(IV.9) are not all independent. After some algebra, one finds that they reduce to just four independent

equations

$$\begin{aligned}\ddot{f} + \lambda f + 2\frac{X_t}{X}\dot{t}\dot{f} &= 0, \\ \dot{t}^2 &= X^2\dot{f}^2 + \lambda Y^2 g^2, \\ X^2 f \dot{f} &= Y^2 g \dot{g}, \\ X^2 \dot{f}^2 + \lambda Y^2 g^2 &= Y^2 \dot{g}^2 + \lambda X^2 f^2.\end{aligned}\tag{4.10}$$

The last two equations of (IV.10) lead to the following two possibilities:

a:

$$\frac{\dot{f}}{f} = \frac{\dot{g}}{g} \quad \text{and} \quad X^2 f \dot{f} = Y^2 g \dot{g},\tag{4.11}$$

which are solved by

$$X(t) = \pm c_1 Y(t), \quad g(\tau) = c_1 f(\tau),\tag{4.12}$$

where c_1 is a constant. After a trivial coordinate redefinition, this corresponds to a circular string in an axially symmetric background.

b:

$$\frac{\dot{f}}{f} = -\lambda \frac{g}{\dot{g}} \quad \text{and} \quad X^2 f \dot{f} = Y^2 g \dot{g},\tag{4.13}$$

from which follows that

$$-\lambda X^2 f^2 = Y^2 \dot{g}^2.\tag{4.14}$$

This equation has no real solutions for tensile strings ($\lambda = 1$), while for null strings ($\lambda = 0$), we find

$$f = \text{const.} \equiv c_1, \quad g = \text{const.} \equiv c_2, \quad t = \text{const.} \equiv c_3,\tag{4.15}$$

with $X(t)$, $Y(t)$ arbitrary. Such solutions, with $t=\text{const.}$, have been considered before in other contexts [14], but since they do not fulfil the physical requirement of forward propagation ($\dot{t} > 0$), we discard them here.

Thus our ansatz (IV.2) eventually only works in the case (IV.12). Then the equations (IV.10) reduce to

$$\ddot{f} + \lambda f + 2\frac{X_t}{X}\dot{t}\dot{f} = 0,\tag{4.16}$$

$$\dot{t}^2 = X^2 \dot{f}^2 + \lambda X^2 f^2. \quad (4.17)$$

For the null strings ($\lambda = 0$), they are immediately solved by

$$\tau = \frac{1}{|c_1|} \int_0^t X(t) dt, \quad (4.18)$$

$$f(t) = c_1 \int_{t_0}^t \frac{dt}{X^2(t)}. \quad (4.19)$$

For the tensile strings ($\lambda = 1$), they can not be solved in general. However, the same equations appeared in a study of strings in Friedmann-Robertson-Walker Universes [14], and some special solutions were found there.

Here we are interested in strings in Bianchi universes. Usually, one starts with the Kasner-type vacuum power-law solutions [15], which are given by

$$X(t) = t^{p_1}, \quad (4.20)$$

$$Y(t) = t^{p_2}, \quad (4.21)$$

$$Z(t) = t^{p_3}, \quad (4.22)$$

and

$$p_1 + p_2 + p_3 = 1 \quad p_1^2 + p_2^2 + p_3^2 = 1, \quad (4.23)$$

where

$$-\frac{1}{3} \leq p_1 \leq 0, \quad 0 \leq p_2 \leq \frac{2}{3}, \quad \frac{2}{3} \leq p_3 \leq 1. \quad (4.24)$$

However, as we saw before, our string ansatz only works in axially symmetric cases. Furthermore, we shall usually also allow the presence of matter.

A special case of the model (IV.1) is an axially symmetric Kasner model in which the matter is that of the stiff-fluid. The metric reads

$$ds^2 = dt^2 - A^2(t) (dx^2 + dy^2) - Z^2(t) dz^2, \quad (4.25)$$

and it is just Kantowski-Sachs metric (III.1) of zero curvature. For the stiff-fluid, $p = \rho$, the conservation law is given by $\rho Z^2 A^4 = \rho t^2 = k^2/16\pi$, which gives the solutions for the scale factors in the form [16]

$$A(t) = t^{p_A}, \quad (4.26)$$

$$Z(t) = t^{1-2p_A}, \quad (4.27)$$

where

$$p_A = \frac{1}{3}[1 \pm \sqrt{1 - 3k^2/2}], \quad (4.28)$$

that is, $0 \leq p_A \leq 2/3$.

Under the ansatz (IV.2) with $f = g$, the equations of motion (IV.16)-(IV.17) become

$$\ddot{f} + \lambda f + \frac{2p_A}{t}\dot{f} = 0, \quad (4.29)$$

$$t^2 - t^{2p_A}(\dot{f}^2 + \lambda f^2) = 0. \quad (4.30)$$

4.1 Tensile strings.

For the tensile strings ($\lambda = 1$), equations (IV.29)-(IV.30) were considered in [14]. They do not seem to be integrable, but some special solutions were found

$$t(\tau) = A \exp(c_1\tau), \quad f(\tau) = B \exp(c_2\tau), \quad (4.31)$$

where the constants (A, B, c_1, c_2) are given by

$$c_1 = \frac{\mp 1}{\sqrt{(p_A - 1)(p_A + 1)}}, \quad c_2 = \pm \sqrt{\frac{p_A - 1}{p_A + 1}}, \quad B = \frac{A^{1-p_A}}{\sqrt{2p_A(p_A - 1)}}. \quad (4.32)$$

However, this solution is not real for the values allowed in our case ($0 \leq p_A \leq 2/3$), so it must be discarded.

All we can do then is to determine the asymptotics of the solutions to equations (IV.29)-(IV.30). One finds for $\tau \rightarrow \infty$

$$t(\tau) = A\tau, \quad (4.33)$$

$$f(\tau) = A^{1-p_A}\tau^{-p_A} \cos \tau, \quad (4.34)$$

where A is an arbitrary positive constant. The invariant string size reads as ($\tau \rightarrow \infty$)

$$S(\tau) = 2\pi A |\cos \tau|, \quad (4.35)$$

so it asymptotically oscillates with constant amplitude and unit frequency, while the comoving string size goes to zero.

4.2 Null strings.

For the null strings ($\lambda = 0$) in axially symmetric Kasner spacetime, the equations (IV.18)-(IV.19) are integrated to give

$$t(\tau) = [|c_1|(1 + p_A)\tau]^{\frac{1}{1+p_A}}, \quad (4.36)$$

$$f(\tau) = c_1 \left(\frac{1 + p_A}{1 - p_A} \right) [|c_1|(1 + p_A)\tau]^{\frac{1-p_A}{1+p_A}}. \quad (4.37)$$

In this case the invariant string size is

$$S(\tau) = 2\pi|c_1| \left(\frac{1 + p_A}{1 - p_A} \right) [|c_1|(1 + p_A)\tau]^{\frac{1}{1+p_A}}, \quad (4.38)$$

which blows up for $\tau \rightarrow \infty$. This is also the case for the comoving string size.

We close this section with some comments on the possibility of having elliptic-shaped strings in anisotropic Bianchi backgrounds. As we saw, the ansatz (IV.2) led to inconsistencies unless $f(\tau) = g(\tau)$ and $X(t) = Y(t)$. However, this does not mean that we must completely rule out the possibility of having elliptic-shaped string configurations. In fact, it is possible to make an ansatz more general than (IV.2), but still describing an elliptic-shaped string. This can be done along the lines of the procedure used in reference [17] (in a somewhat different context): We discard the orthonormal gauge and work directly with the Nambu-Goto action. In that case, the ansatz (IV.2) leaves us more freedom than before. Unfortunately, the equations of motion now become more complicated than before, but at least they are not explicitly inconsistent, and there is some hope that one can find special solutions or at least solve the equations numerically.

In the orthonormal gauge (II.2)-(II.3), as used in this paper, this more general ansatz corresponds to replacing (IV.2) by

$$X^0 = t(\tau), \quad X^1 = x = f(\tau) \cos \phi(\tau, \sigma), \quad X^2 = y = g(\tau) \sin \phi(\tau, \sigma), \quad X^3 = z = \text{const.}, \quad (4.39)$$

and the function $\phi(\tau, \sigma)$ gives us the extra freedom as mentioned above. However, we leave the implications of using the ansatz (IV.39) for investigations elsewhere.

5 Strings in Axisymmetric Bianchi type IX Background

Another interesting example of a curved background for strings, we consider, is the Bianchi type IX background. It generalizes the $k = +1$ isotropic Friedmann model to the case of anisotropic spacetimes. In order to show the relation, it has been shown, among others, that all BIX models recollapse similarly as $k = +1$ Friedmann models [18]. The general case cannot be solved analytically for the scale factors and they subject to chaotic behaviour. It would be interesting to find out whether the test strings in such a general background also behave chaotically, but for the moment we leave this question for a separate paper, and consider just an axially symmetric Bianchi IX model which can be solved analytically. The metric of such a model in a holonomic frame, is given by [19, 20]

$$ds^2 = dt^2 - c^2(t) (d\psi + \cos \theta d\varphi)^2 - a^2(t) (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (5.1)$$

where ψ, θ, φ are the Euler angles ($0 \leq \psi \leq 4\pi, 0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$). We use the following ansatz for the spacetime coordinates

$$X^0 = t(\tau), \quad X^1 = \psi(\tau), \quad X^2 = \theta(\tau), \quad X^3 = \varphi = \sigma, \quad (5.2)$$

and the equations of motion (II.1) then read

$$\ddot{t} + cc_{,t}\dot{\psi}^2 + aa_{,t}\dot{\theta}^2 - \lambda (cc_{,t} \cos^2 \theta + aa_{,t} \sin^2 \theta) = 0, \quad (5.3)$$

$$\ddot{\psi} + 2\frac{c_{,t}}{c}t\dot{\psi} + \frac{c^2}{a^2} \cot \theta \dot{\psi} \dot{\theta} = 0, \quad (5.4)$$

$$\ddot{\theta} + 2\frac{a_{,t}}{a}t\dot{\theta} - \lambda \frac{c^2 - a^2}{a^2} \sin \theta \cos \theta = 0, \quad (5.5)$$

$$\frac{c^2}{a^2} \frac{1}{\sin \theta} \dot{\psi} \dot{\theta} = 0. \quad (5.6)$$

The last of these equations (V.6) says that either $\psi = \text{const.}$ or $\theta = \text{const.}$ ($\theta \neq 0$). The constraints (II.2)-(II.3) read

$$\ddot{t}^2 - c^2 \dot{\psi}^2 - a^2 \dot{\theta}^2 - \lambda (a^2 \sin^2 \theta + c^2 \cos^2 \theta) = 0, \quad (5.7)$$

$$c^2 \cos \theta \dot{\psi} = 0, \quad (5.8)$$

from which it follows that either ψ must be constant or $\theta = \pi/2$. We will consider both cases. For each of these cases the invariant string size is given by

$$S(\tau) = 2\pi\sqrt{a^2(t(\tau))\sin^2\theta(\tau) + c^2(t(\tau))\cos^2\theta(\tau)}. \quad (5.9)$$

The well-known stiff-fluid solutions of the Einstein equations for Bianchi IX axisymmetric model are given by [20]

$$c^2 = \frac{A}{\cosh A\eta}, \quad (5.10)$$

$$a^2 = \frac{B^2 \cosh A\eta}{4A \cosh^2(\frac{B}{2}\eta)}, \quad (5.11)$$

$$p = \rho = \frac{M^2}{4a^4c^2}, \quad (5.12)$$

where A, B, M are constants ($A > |B|$) with

$$B^2 = A^2 - M^2, \quad (5.13)$$

$$t = \int^\eta a^2|c|d\eta. \quad (5.14)$$

One can easily see that the vacuum solution $M = 0$ is given by

$$c^2 = \frac{A}{\cosh A\eta}, \quad (5.15)$$

$$a^2 = \frac{A \cosh A\eta}{4 \cosh^2(\frac{A}{2}\eta)}. \quad (5.16)$$

Notice that the scale factor $c(\eta)$ increases from $c(-\infty) = 0$ to $c(0) = c_{\max} = \sqrt{A}$ and then decreases to $c(\infty) = 0$ again. On the other hand, the scalefactor $a(\eta)$ decreases from $a(-\infty) = \sqrt{A/2}$ to $a(0) = a_{\min} = \sqrt{A/4}$ and then increases to $a(\infty) = \sqrt{A/2}$ again. However, the volume essentially follows the scalefactor $c(\eta)$, i.e., the universe is of recollapsing type.

5.1 $\theta = \pi/2$ solutions

If we assume $\theta = \pi/2$ and $\dot{\psi} \neq 0$, then we can easily integrate (V.4) to give

$$\dot{\psi} = \frac{m}{c^2}, \quad (5.17)$$

with $m = \text{const}$. Then we have

$$\tau(t) = \int_0^t \frac{|c|dt}{\sqrt{m^2 + \lambda a^2 c^2}}, \quad (5.18)$$

$$\psi(t) = m \int_{t_0}^t \frac{dt}{|c|\sqrt{m^2 + \lambda a^2 c^2}}. \quad (5.19)$$

For the null strings ($\lambda = 0$), we can easily get the solution for the vacuum case $M = 0$ in terms of parametric time η , by using (V.15) and (V.16), i.e.,

$$d\eta = \frac{|m|}{a^2 c^2} d\tau, \quad (5.20)$$

which is integrated to

$$\eta(\tau) = \frac{2}{A} \operatorname{arcth} \left(\frac{2|m|}{A} \tau \right), \quad (5.21)$$

where we took boundary conditions such that $\eta(0) = 0$. The solutions for the scale factors $c(\tau)$ and $a(\tau)$ are given by

$$c^2(\tau) = A \left(\frac{A^2 - 4m^2 \tau^2}{A^2 + 4m^2 \tau^2} \right), \quad (5.22)$$

$$a^2(\tau) = \frac{1}{4A} (A^2 + 4m^2 \tau^2), \quad (5.23)$$

and $\tau^2 \leq A^2/4m^2$. Then we find

$$t(\tau) = \frac{|m|}{\sqrt{A}} \int_0^\tau d\tau \sqrt{\frac{A^2 + 4m^2 \tau^2}{A^2 - 4m^2 \tau^2}}, \quad (5.24)$$

$$\psi(\tau) = \psi_0 + \frac{m}{A} \int_0^\tau d\tau \left(\frac{A^2 + 4m^2 \tau^2}{A^2 - 4m^2 \tau^2} \right). \quad (5.25)$$

The integral for $t(\tau)$ is of elliptic type, while the one for $\psi(\tau)$ is elementary. However, we shall not need the explicit results here.

The invariant string size reads

$$S(\tau) = \frac{\pi}{\sqrt{A}} \sqrt{A^2 + 4m^2 \tau^2}. \quad (5.26)$$

From the above, we conclude that for the admissible values of the parameter τ , the string starts with the size $S = \pi\sqrt{2A}$ for $\tau = -A/2|m|$, then contracts to the size $S = \pi\sqrt{A}$ for $\tau = 0$, and expands again to $S = \pi\sqrt{2A}$ for $\tau = A/2|m|$. This can be easily understood physically since, for $\theta = \pi/2$, the string is winding in the φ -direction with scale factor a .

5.2 The case $\dot{\psi} = 0, \dot{\theta} \neq 0$

In this case, the equations (V.3)-(V.8) become

$$\ddot{t} + aa_{,t}\dot{\theta}^2 - \lambda \left(cc_{,t} \cos^2 \theta + aa_{,t} \sin^2 \theta \right) = 0, \quad (5.27)$$

$$\ddot{\theta} + 2\frac{a_{,t}}{a}\dot{t}\dot{\theta} - \lambda \frac{c^2 - a^2}{a^2} \sin \theta \cos \theta = 0, \quad (5.28)$$

$$\dot{t}^2 - a^2\dot{\theta}^2 - \lambda \left(a^2 \sin^2 \theta + c^2 \cos^2 \theta \right) = 0. \quad (5.29)$$

Notice that the first equation can be obtained from the two others. Thus we have just two coupled equations; one of first order and one of second order. For the tensile strings, the general solution does not seem to be available. For the null strings, we can integrate for $\theta(\tau)$ and $t(\tau)$

$$\dot{\theta} = \frac{s}{a^2}, \quad (5.30)$$

$$\dot{t}^2 = \frac{s^2}{a^2}, \quad (5.31)$$

with $s = \text{const}$. Then using the exact vacuum solutions (V.15)-(V.16), we can integrate this further since $(d\eta/d\tau)^2 = s^2/a^6c^2$. One finds

$$|s|d\tau = \frac{A^2}{8} \frac{\cosh A\eta}{\cosh^3 \left(\frac{A}{2}\eta \right)} d\eta, \quad (5.32)$$

which can be integrated explicitly in terms of elementary functions. In principle we can then also obtain expressions for $t(\tau)$ and $\theta(\tau)$. However, it turns out to be somewhat simpler to express everything in terms of the parametric time η . For instance

$$d\theta = \frac{s}{|s|} |ac| d\eta, \quad (5.33)$$

which leads to

$$\theta(\eta) - \theta_0 = 2 \frac{s}{|s|} \text{arctg} \left(e^{A\eta/2} \right). \quad (5.34)$$

It is then straightforward to write down an explicit expression for the invariant string size similar to equation (V.9), but with S as a function of η , since the scale factors are already given in terms of η .

It follows from the above results that during the whole evolution of the Universe, the polar angle θ changes by π . Thus there are two scenarios: If

$\theta_0 = 0$, then the string starts with zero size at one of the poles, then expands and eventually collapses to zero size again at the other pole. On the other hand, if $\theta_0 \neq 0$, then the string starts with finite size, passes one of the poles (still with finite size) and eventually ends up with a finite size. Thus the behaviour is qualitatively similar to that of strings in Kantowski-Sachs spacetimes as described in Section III.

6 Summary

In this paper we have considered the tensile and null string evolution and propagation in some homogeneous but anisotropic spacetimes of Kantowski-Sachs and Bianchi type. This generalizes and completes earlier investigations of strings in more symmetric backgrounds.

Our results demonstrate the richness of different evolution schemes for extended objects, here strings, in curved backgrounds. For the tensile strings, this is due to the "competition" between the string tension and the gravitational field, which together determine the evolution of the string. For the null strings, it is simply due to the fact that we are dealing with an extended object in a gravitational field, i.e., the object subjects to tidal forces. In both cases, the situation should be compared with the conceptually much simpler problems of point particle propagation in curved spacetimes and string evolution and propagation in flat Minkowski spacetime.

We mainly considered closed circular strings, which allowed us to obtain simple exact analytical results in most cases. We essentially saw three qualitatively different kinds of circular string evolution: a) the string simply follows the expansion or contraction of the Universe, b) the string makes a finite or infinite number of oscillations during the evolution of the Universe, c) the contraction of the string is exactly balanced by the expansion of the Universe, such that the physical string size is constant.

We also discussed the problems of obtaining consistent equations of motion describing an elliptic-shaped string configuration. The use of the simplest, and a priori most natural ansatz, describing an elliptic-shaped string, led to inconsistent equations of motion. At this point, we leave for future work the question of whether a more complicated ansatz, as described at the end of Section IV, with its more complicated equations of motion, can solve these problems.

7 Acknowledgments

We wish to thank A.A.Zheltukhin for useful discussions. MPD is supported by the Polish Research Committee (KBN) grant No 2 PO3B 196 10. The work of ALL was partly supported by NSERC (Canada) and CNRS (France).

References

- [1] H. J. de Vega and N. Sánchez, in *Advances in Astrofundamental Physics* (proceedings) ed. by N. Sánchez (World Scientific, 1994) 227; A.L. Larsen and N. Sánchez, in *Advances in Astrofundamental Physics* (proceedings) ed. by N. Sánchez (World Scientific, 1994) 272.
- [2] A. Schild, Phys. Rev. D**16**, 1722 (1977); A. Karlhede and U. Lindström, Class. Quantum Grav. **3**, L73 (1986).
- [3] S. Kar, Phys. Rev. D**53**, 6842 (1996); S.N. Roshchupkin and A.A. Zheltukhin, Classical Quant. Grav. **12**, 2519 (1995); M.P. Dąbrowski and A.L. Larsen, Phys. Rev. D**55**, 6409 (1997).
- [4] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. B**261**, 1 (1985); C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. B**262**, 593 (1985).
- [5] H.J. De Vega and A. Nicolaidis, Phys. Lett. B**295**, 214 (1992); H.J. De Vega, I. Giannakis and A. Nicolaidis, Mod. Phys. Lett. A**10**, 2479 (1995); C.O. Lousto and N. Sanchez, Phys. Rev. D**54**, 6399 (1996); A.A. Zheltukhin, Class. Quantum Grav. **13**, 2357 (1996).
- [6] J. Mimóso and D. Wands, Phys. Rev. D**51**, 477 (1995); J.D. Barrow and M.P. Dąbrowski, Phys. Rev. D**55**, 630 (1997).
- [7] R. Kantowski and R. K. Sachs, J. Math. Phys. **7**, 443 (1966).
- [8] Ø. Grøn, Journ. Math. Phys. **27**, 1490 (1986).
- [9] M.P. Dąbrowski, Astrophys. Space Sci. **240**, 123 (1996).
- [10] C.B. Collins, J. Math. Phys. **18**, 2116 (1977).

- [11] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).
- [12] M. Mueller, Nucl. Phys. **B37**, 37 (1990); E.J. Copeland, A. Lahiri and D. Wands, Phys. Rev. **D50**, 4868 (1994); E.J. Copeland, A. Lahiri and D. Wands, Phys. Rev. **D51**, 1569 (1995); N.A. Batakis and A. Kehagias, Nucl. Phys. **B449**, 248, (1995); M. Gasperini and R. Ricci, Class. Quantum Grav. **12**, 677 (1995); J.D. Barrow and K.E. Kunze, Phys. Rev. **D55**, 623 (1997).
- [13] M.P. Ryan and L.C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton U.P., Princeton, 1975).
- [14] H.J. de Vega and I.L. Egusquiza, Phys. Rev. **D49** 763 (1994).
- [15] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, 1975).
- [16] K.S. Thorne, Astrophys. Journ. **148**, 51 (1967).
- [17] X. Li and H. Cheng, Class. Quantum Grav. **13**, 225 (1996).
- [18] X.-F. Lin and R.M. Wald, Phys. Rev. **D40**, 3280 (1989); Phys. Rev. **D41**, 2444 (1990).
- [19] D.R. Brill, Phys. Rev. **B133**, 845 (1964).
- [20] J.D. Barrow, Nature **272**, 211 (1978); Nucl. Phys. **B296**, 697 (1988).